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Asymptotic Expansions for a Class of Elliptic Difference Schemes*

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Abstract. In this paper, we derive an asymptotic expansion of the global error for Kreiss' difference scheme for the Dirichlet problem for Poisson's equation. This scheme, combined with a deferred correction procedure or the Richardson extrapolation technique, yields a method of accuracy at least $O(h^{6.5})$ in L_2, where h is the mesh length.

1. Introduction. In Section 2 of this paper we consider a family of difference schemes for the Dirichlet problem for Poisson's equation in n dimensions. The schemes are based on the standard $(2n + 1)$-point formula combined with polynomial extrapolation formulas of high degree, k say, at the boundary. Kreiss has developed an interesting method for proving the convergence of schemes of this kind, by reducing the stability investigations to one-dimensional problems. In a recent paper by Pereyra, Proskurowski, and Widlund [2], the stability has been proved, for $1 \leqslant k \leqslant 6$, by using Kreiss' method. In the paper [2], it is also proved that, for $k = 6$, there exists an asymptotic expansion of the global error of the form

$$v = u + h^2 e_2 + h^4 e_4 + r_h, \qquad \|r_h\|_2 = O(h^{5.5}),$$

where v and u are the solutions to the discrete and the continuous problems, respectively, h is the mesh length, e_2 and e_4 are smooth functions independent of h, and $\|\cdot\|_2$ is the usual discrete n-dimensional L_2-norm. The main result of Section 2 is the following extension of the above expansion

$$v = u + h^2 e_2 + h^4 e_4 + h^6 e_6 + r_h, \qquad \|r_h\|_2 = O(h^{6.5}), \tag{1.1}$$

which is obtained by a refined stability investigation with respect to the inhomogeneous term in the boundary condition. By using three or four different mesh lengths, (1.1) guarantees that we get an error of order $O(h^6)$ or $O(h^{6.5})$, respectively, by the Richardson extrapolation method. A deferred correction method is very likely less costly to use since it only requires one mesh length; see [1]. For a description of the latter method and for several numerical experiments see [2].

Finally, we point out that the kind of meshes used in this paper are not suitable for Neumann problems, for which we instead suggest the use of composite mesh methods; see [3] and [4].

2. An Asymptotic Expansion of the Global Error for Kreiss' Method. We begin this section with a brief account of Kreiss' difference scheme for the Dirichlet problem for Poisson's equation. Almost the same notations will be used as in [2],

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where also a more thorough description of the method can be found. The continuous problem is denoted by

$$(2.1)\qquad \begin{aligned} -\sum_{i=1}^{n} \partial^2 u/\partial x_i^2 &= f(x), \qquad x \in \Omega,\\ u(x) &= g(x), \qquad x \in \partial\Omega, \end{aligned}$$

where the region Ω is an open, bounded subset of the n-dimensional, real Euclidean space R^n with the smooth boundary $\partial\Omega$. The smoothness requirements needed for the solution will be apparent later.

A uniform grid R_h^n is defined by

$$R_h^n = \{x \in R^n \mid x_i = x_i^{(0)} + n_i h, n_i = 0, \pm 1, \pm 2, \dots\},$$

where $h > 0$ is the mesh length and $(x_1^{(0)}, x_2^{(0)}, \dots, x_n^{(0)})$ is a fixed point in R^n. Let $\Omega_h = \Omega \cap R_h^n$ and define Ω_h^* to be the set of gridpoints $x \in \Omega_h$ such that at least one of the points $x \pm he_i$, $i = 1, 2, \dots, n$, is not in Ω_h, where the vector e_i is the unit vector in the direction of the positive ith coordinate axis. The points in Ω_h^* are called irregular gridpoints. For each $x \in \Omega_h$, we initially apply the second-order difference approximation

$$(2.2)\qquad 2nv(x) - \sum_{i=1}^{n} (v(x - he_i) + v(x + he_i)) = h^2 f(x).$$

For an irregular gridpoint x, this formula is modified in the following way. Assume that $x - he_i \notin \Omega_h$. Then $v(x - he_i)$ shall be eliminated from (2.2) by using a polynomial extrapolation formula of a fixed degree k

$$(2.3)\qquad \begin{aligned} -v(x - he_i) &= \sum_{j=1}^{k} \beta_j v(x + h(j-1)e_i) - \frac{1}{\alpha_0} g(x^*),\\ \beta_j &= (-1)^{j+1} \frac{s}{j-s} \binom{k}{j}, \qquad j = 1, 2, \dots, k,\\ \alpha_0 &= (1-s)(2-s)(3-s) \cdots (k-s)/k!, \end{aligned}$$

where x^* is the intersection of $\partial\Omega$ and the line segment between $x - he_i$ and x and hence $x^* = x - he_i + she_i$, where $0 \leqslant s < 1$. It is now easily seen that the coefficient matrix A of the difference scheme can be written as

$$A = \sum_{i=1}^{n} P_i^T A_i P_i,$$

where the matrices A_i correspond to diffrences in the ith coordinate direction and are the direct sum of matrices of the form

$$\begin{bmatrix} (2+\beta_1) & (-1+\beta_2) & \beta_3, \dots, \beta_k & & & \\ -1 & 2 & -1 \dots & & & \\ 0 & -1 & 2 & & & \\ \dots & & & \dots 2 & -1 & 0 \\ & & & \dots -1 & 2 & -1 \\ & & & \tilde\beta_k, \dots, \tilde\beta_3 & (-1+\tilde\beta_2) & (2+\tilde\beta_1) \end{bmatrix}.$$

The matrices P_i are permutation matrices corresponding to different orderings of the gridpoints.

In [2] it was proved that, for $1 \leqslant k \leqslant 6$, there is a constant C, independent of h, such that

$$w^T B w \geqslant Ch^2/(\text{diameter}(\Omega))^2 \cdot w^T w, \tag{2.4}$$

for all vectors w with dimension equal to the order of B. Since A_i is a direct sum of matrices of the type B, it immediately follows that (2.4) is valid with B replaced by A_i. It also immediately follows that

$$v^T A v \geqslant nCh^2/(\text{diameter}(\Omega))^2 \cdot v^T v, \tag{2.5}$$

for all vectors v, which implies that

$$\|A^{-1}\| \leqslant \frac{(\text{diameter}(\Omega))^2}{nCh^2}, \tag{2.6}$$

where the spectral matrix norm has been used. By using this estimate it was proved in [2] that

$$\begin{gathered} v = u + h^2 e_2 + h^4 e_4 + r_h, \\ \|r_h\|_2 = \Big(\sum_{x \in \Omega_h} |r_h(x)|^2 h^n\Big)^{1/2} \leqslant O(h^{5.5}), \end{gathered} \tag{2.7}$$

where e_2 and e_4 are smooth functions independent of h. In order to get a more complete asymptotic expansion for the global error, we need a sharper stability result, with respect to the inhomogeneous term in the boundary condition, than the one that follows from (2.6).

Let $[\Omega_h^*]$ denote the set of grid functions y defined on Ω_h with $y(x) = 0$ for $x \notin \Omega_h^*$. We shall now prove that, for $1 \leqslant k \leqslant 6$, there is a constant C_1, independent of h, such that

$$v^T A v \geqslant nC_1 h/\text{diameter}(\Omega) \cdot v^T v, \quad \text{for } Av = y \in [\Omega_h^*]. \tag{2.8}$$

From this estimate it immediately follows that

$$\|A^{-1} y\| \leqslant \text{diameter}(\Omega)/(nC_1 h) \cdot \|y\|, \qquad y \in [\Omega_h^*]. \tag{2.9}$$

We shall now prove (2.8) by first proving a similar inequality for the matrices of the type B. Let us consider the system of linear equations

$$Bw = \begin{bmatrix} g_0/\alpha_0 \\ 0 \\ \vdots \\ 0 \\ g_N/\tilde{\alpha}_0 \end{bmatrix}, \tag{2.10}$$

which is a discretization of the one-dimensional problem $-z'' = 0$, $z(0) = g_0$, $z(a) = g_N$, where a is a positive constant and $z(x) = g_0(a - x)/a + g_N x/a$. Let us introduce the gridpoints $x_\nu = x_0 + \nu h$, $\nu = 0, 1, 2, \ldots, N + 1$, where N is the order of the matrix B. Further $-x_0 = sh$ and $x_{N+1} - a = \tilde{s}h$, where s and $\tilde{s}$ are the quantities appearing in α_0 and $\tilde{\alpha}_0$, respectively. The system (2.10) can now be written as

$$-w_{\nu-1} + 2w_\nu - w_{\nu+1} = 0, \qquad \nu = 1, 2, \ldots, N,$$

$$-w_0 = \sum_{j=1}^{k} \beta_i w_j - \frac{1}{\alpha_0} g_0, \qquad -w_{N+1} = \sum_{j=1}^{k} \tilde{\beta}_j w_{N+1-j} - \frac{1}{\tilde{\alpha}_0} g_N.$$

Since z is a linear function, $w_\nu = z(x_\nu)$, for $k \geqslant 1$, i.e.

$$w_\nu = g_0(a - x_\nu)/a + g_N x_\nu / a. \tag{2.11}$$

From (2.10) and from the above expression for w_ν, we get

$$\begin{aligned} a w^T B w &= g_0^2 \frac{a - x_1}{\alpha_0} + g_0 g_N \left(\frac{x_1}{\alpha_0} + \frac{a - x_\nu}{\tilde{\alpha}_0} \right) + g_N^2 \frac{x_N}{\tilde{\alpha}_0} \\ &\geqslant g_0^2 \left(\frac{a - x_1}{\alpha_0} - \frac{1}{2}\left(\frac{x_1}{\alpha_0} + \frac{a - x_N}{\tilde{\alpha}_0} \right) \right) + g_N^2 \left(\frac{x_N}{\tilde{\alpha}_0} - \frac{1}{2}\left(\frac{x_1}{\alpha_0} + \frac{a - x_N}{\tilde{\alpha}_0} \right) \right). \end{aligned}$$

Since $a - x_1 \geqslant (N-1)h$, $x_N \geqslant (N-1)h$, $x_1 = (1-s)h$, and $a - x_N = (1 - \tilde{s})h$ and further $0 < \alpha_0, \tilde{\alpha}_0 \leqslant 1$, $\alpha_0/(1-s) \geqslant 1/k$, and $\tilde{\alpha}_0/(1-\tilde{s}) \geqslant 1/k$, we get

$$w^T B w \geqslant \frac{(N-1)h - kh}{a} (g_0^2 + g_N^2). \tag{2.12}$$

Let us now consider the quantity $w^T w$ which, according to (2.11), can be written as

$$\begin{aligned} w^T w &= \frac{1}{h} \left(g_0^2 \sum_{\nu=1}^{N} \left(\frac{a - x_\nu}{a} \right)^2 h + 2 g_0 g_N \sum_{\nu=1}^{N} \frac{x_\nu}{a} \left(\frac{a - x_\nu}{a} \right) h + g_N^2 \sum_{\nu=1}^{N} \left(\frac{x_\nu}{a} \right)^2 h \right) \\ &\leqslant \frac{2}{h} \max \left(\sum_{\nu=1}^{N} \left(\frac{a - x_\nu}{a} \right)^2 h, \sum_{\nu=1}^{N} \left(\frac{x_\nu}{a} \right)^2 h \right) (g_0^2 + g_N^2) \\ &\leqslant \frac{2((N+1)h)^3}{3a^2 h} (g_0^2 + g_N^2). \end{aligned}$$

By using (2.12) and the above inequality, we easily get

$$w^T B w \geqslant 3h(1 - (k+2)/(N+1))/(2a) \cdot w^T w,$$

where we also have used that $(N+1)h \geqslant a$. For later references we write this inequality as

$$w^T B w \geqslant h C_1 / \text{diameter}(\Omega) \cdot w^T w, \quad C_1 = 3/(2(k+3)), \quad N > k+1. \tag{2.13}$$

Note that (2.13) is valid only for w satisfying (2.10). The inequality (2.8) can now be obtained in the same way as (2.5).

Let us for functions $y \in [\Omega_h^*]$ define the following $n-1$-dimensional L_2-norm

$$|y|_2 = \left(\sum_{x \in \Omega_h^*} |y(x)|^2 h^{n-1} \right)^{1/2}.$$

We can now write (2.9) in the following way

$$Av = y \in [\Omega_h^*] \Rightarrow \|v\|_2 \leqslant \text{diameter}(\Omega) / \left(n C_1 \sqrt{h} \right) \cdot |y|_2, \tag{2.14}$$

where $\|\cdot\|_2$ is the norm defined in (2.7). For later use we also write down the local truncation error to the extrapolation formula (2.3)

$$\frac{(-1)^k}{k+1} sh^{k+1} u^{(k+1)}. \tag{2.15}$$

We shall now derive the improved version of the asymptotic expansion of the global discretization error and consider for definiteness the case $k = 6$. We make the Ansatz

$$v = u + h^2 e_2 + h^4 e_4 + h^6 e_6 + r_h,$$

where e_2, e_4, and e_6 are smooth functions, independent of h, satisfying the boundary condition $e_t = 0$, on $\partial\Omega$, $t = 2, 4, 6$. We shall prove that $\|r_h\|_2 = O(h^{6.5})$. For the solution u of (2.1), we have

$$Au = h^2 f + G + h^4 l_4(u) + h^6 l_6(u) + h^8 l_8(u) + O(h^7)G_1 + O(h^9), \tag{2.16}$$

where the l_t are differential operators of order t with constant coefficients, $t = 2, 4, 6$, and 8, G and $O(h^7)G_1$ belong to $[\Omega_h^*]$ and correspond to the inhomogeneous boundary condition and to (2.15), respectively. We note that the difference scheme is given by $Av = h^2 f + G$ and further that

$$Ae_t = h^2 Le_t + h^4 l_4(e_t) + h^6 l_6(e_t) + O(h^7), \qquad t = 2, 4, 6, \tag{2.17}$$

where L is the differential operator defined in (2.1). By multiplying the Ansatz for v by A and by using (2.16) and (2.17), we get that

$$\begin{aligned} Av &= Au + \sum_{t=1}^{3} h^{2t} Ae_{2t} + Ar_h \\ &= h^2 f + G + h^4 l_4(u) + h^6 l_6(u) + h^8 l_8(u) + O(h^7)G_1 + O(h^9) \\ &\quad + h^4 Le_2 + h^6 l_4(e_2) + h^8 l_6(e_2) + O(h^9) + h^6 Le_4 + h^8 l_4(e_4) \\ &\quad + O(h^{10}) + h^8 Le_6 + O(h^{10}) + Ar_h = h^2 f + G. \end{aligned}$$

By determining e_2, e_4, and e_6 by

$$Le_2 + l_4(u) = 0, \qquad Le_4 + l_4(e_2) + l_6(u) = 0,$$
$$Le_6 + l_4(e_4) + l_6(e_2) + l_8(u) = 0, \qquad e_t = 0 \quad \text{on } \partial\Omega, \qquad t = 2, 4, 6,$$

we get

$$Ar_h = -G_1 O(h^7) + O(h^9).$$

Since $G_1 \in [\Omega_h^*]$ and $|G_1|_2 = O(1)$, it follows from (2.14) and (2.6) that

$$\|r_h\|_2 = O(h^{6.5}), \tag{2.18}$$

which is the main result of this paper.

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